

COMMUTATIVITY UP TO A FACTOR FOR BOUNDED AND UNBOUNDED OPERATORS

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ABSTRACT. In this paper, we further investigate the problem of commutativity up to a factor (or λ -commutativity) in the setting of bounded and unbounded linear operators in a complex Hilbert space. The results are based on a new approach to the problem. We finish the paper by a conjecture on the commutativity of self-adjoint operators.

1. INTRODUCTION

Commutation relations between self-adjoint operators in a complex Hilbert space are important in the interpretation of quantum mechanical observables. They also play an important role when analyzing their spectra. For more details see [1] (and the references therein), [4] and [16].

Recently, commutativity up to a factor has been given much attention by many authors. See e.g. [1], [2], [8], [11] and [19].

The purpose of the present paper is twofold. First, we recover known results by examining bounded normal products of self-adjoint operators. Second, we extend this method to unbounded operators.

Let us say a little more about details of this technique. It is well-known that two bounded, normal and commuting operators have a normal product. The proof uses the celebrated Fuglede theorem (we note that this question has been generalized to the case of unbounded operators in e.g. [7], [13], [14] and [15]). In a very similar manner, we also notice that -this time via the Fuglede-Putnam theorem- the product of two anti-commuting normal operators remains normal. So, we conjectured that the product of normal operators which commute up to a factor would be normal. This is in effect the case and the reason why we want to use the normality of the product in question is that we may exploit results on the bounded normal product of self-adjoint operators (as those in [9] and [10], and the references therein). The advantage of this approach is that it also extends to unbounded operators so that we may again take advantage of the results in [9] and [10].

To make the paper as self-contained as possible, we recall the following results:

Theorem 1.1. *Let A be a densely defined unbounded operator.*

- (1) $(BA)^* = A^*B^*$ if B is bounded.
- (2) $A^*B^* \subset (BA)^*$ for any densely unbounded B and if BA is densely defined.
- (3) Both AA^* and A^*A are self-adjoint whenever A is closed.

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Lemma 1.1 ([18]). *If A and B are densely defined and A is invertible with inverse A^{-1} in $B(H)$, then $(BA)^* = A^*B^*$.*

Proposition 1.1 ([5]). *Let A , B and C be unbounded self-adjoint operators. Then*

$$A \subseteq BC \implies A = BC.$$

Theorem 1.2. [Fuglede-Putnam theorem, for a proof see e.g. [3]] *If A is a bounded operator and if M and N are normal operators, then*

$$AN \subseteq MA \implies AN^* \subseteq M^*A$$

(if N and M are bounded, then we replace " \subseteq " by " $=$ ").

Theorem 1.3. [9] *Let A and B be two self-adjoint operators such that AB is normal. Then*

- (1) *If A and B are both bounded, and if further $\sigma(A) \cap \sigma(-A) \subseteq \{0\}$ or $\sigma(B) \cap \sigma(-B) \subseteq \{0\}$, then AB is self-adjoint.*
- (2) *If only B is bounded, and if $\sigma(B) \cap \sigma(-B) \subseteq \{0\}$, then AB is self-adjoint.*

Theorem 1.4. [1] *Let A , B be bounded operators such that $AB \neq 0$ and $AB = \lambda BA$, $\lambda \in \mathbb{C}^*$. Then*

- (1) *if A or B is self-adjoint, then $\lambda \in \mathbb{R}$;*
- (2) *if both A and B are self-adjoint, then $\lambda \in \{-1, 1\}$; and*
- (3) *if A and B are self-adjoint and one of them is positive, then $\lambda = 1$.*

Theorem 1.5. [19] *Let A , B be bounded operators such that $AB = \lambda BA \neq 0$, $\lambda \in \mathbb{C}^*$. Then*

- (1) *if A or B is self-adjoint, then $\lambda \in \mathbb{R}$;*
- (2) *if either A or B is self-adjoint and the other is normal, then $\lambda \in \{-1, 1\}$; and*
- (3) *if A and B are both normal, then $|\lambda| = 1$.*

Theorem 1.6. [11] *Let A be an unbounded operator and let B be a bounded one. Assume that $BA \subset \lambda AB \neq 0$ where $\lambda \in \mathbb{C}$. Then*

- (1) *λ is real if A is self-adjoint.*
- (2) *$\lambda = 1$ if $0 \notin W(B)$ (the numerical range of B) and if A is normal; hence $\lambda = 1$ if B is strictly positive and A is normal.*
- (3) *$\lambda \in \{-1, 1\}$ if A is normal and B is self-adjoint.*

In the end, we assume other notions and results on both bounded and unbounded operators (some general textbooks are [3], [17] and [18]). In particular, the reader should be aware that invertible operators are taken to have an everywhere defined bounded inverse, and that if A and B are two densely defined unbounded operators, then

$$(\lambda A)B = A(\lambda B) = \lambda(AB)$$

whenever $\lambda \in \mathbb{C}^*$.

2. MAIN RESULTS

2.1. The Bounded Case.

Definition 2.1. Two bounded operators A and B are said to commute up to a factor λ (or λ -commute) if $AB = \lambda BA$ for some complex λ .

As alluded to in the introduction, we take on the problem of commutativity up to a factor differently, that is, we first prove that the product of two bounded normal λ -commuting operators is normal. We have

Theorem 2.1. *Let A and B be two bounded normal operators such that $AB = \lambda BA \neq 0$ where $\lambda \in \mathbb{C}$. Then AB (and also BA) is normal for any non-zero λ .*

Proof. Since A and B are both normal, so are λA and λB . So by Theorem 1.2, we have

$$AB = \lambda BA \implies AB^* = \overline{\lambda} B^* A \text{ and } A^* B = \overline{\lambda} B A^*.$$

Then we have on the one hand

$$(AB)^* AB = B^* A^* AB = B^* A A^* B = \overline{\lambda} B^* A B A^* = |\lambda|^2 B^* B A A^*.$$

On the other hand we obtain

$$AB(AB)^* = ABB^* A^* = AB^* B A^* = \overline{\lambda} B^* A B A^* = |\lambda|^2 B^* B A A^*.$$

Thus AB is normal. \square

Remark. Of course, thanks to Theorem 1.5, the condition $|\lambda| = 1$ was tacitly assumed in the previous theorem, but we would have not needed it (cf. Theorem 2.2).

Corollary 2.1. *Let A and B be two bounded self-adjoint operators satisfying $AB = \lambda BA \neq 0$ where $\lambda \in \mathbb{C}$. If either $\sigma(A) \cap \sigma(-A) \subseteq \{0\}$ or $\sigma(B) \cap \sigma(-B) \subseteq \{0\}$, then $\lambda = 1$.*

Proof. By Theorem 2.1, AB and BA are normal. By Theorem 1.3, AB and BA are self-adjoint. Hence

$$BA = (AB)^* = AB = \lambda BA,$$

yielding $\lambda = 1$. The proof is thus complete. \square

Corollary 2.2. *Let A and B be two bounded self-adjoint operators satisfying $AB = \lambda BA \neq 0$ where $\lambda \in \mathbb{C}$. Then $\lambda = 1$ if one of the following occurs:*

- (1) A is positive;
- (2) $-A$ is positive;
- (3) B is positive;
- (4) $-B$ is positive.

Corollary 2.3. *If both A and B are self-adjoint (and bounded), then*

$$AB = \lambda BA \implies \lambda \in \{-1, 1\}.$$

Proof. We have

$$A^2 B = \lambda A B A = \lambda^2 B A^2.$$

Since A is self-adjoint, A^2 is positive so that Corollary 2.2 gives $\lambda^2 = 1$ or $\lambda \in \{-1, 1\}$. \square

2.2. A digression.

Definition 2.2. Two bounded operators A and B are said to **commute unitarily** if $AB = UBA$ for some unitary operator U .

Proposition 2.1. *Let A and B be two bounded normal operators such that $AB = UBA$ for some unitary operator U . If U commutes with B , then UB and AB are both normal.*

We omit the proof as it is very similar to that of Theorem 2.1 and hence we leave it to the interested reader.

2.3. The Unbounded Case. We may split the main result in this subsection into two parts for the first one of the two is important in its own right. It also generalizes known results for two normal operators (where at least one of them is bounded, see e.g. [7] and [13]).

Theorem 2.2. *Let A and B be two normal operators where B is bounded. Assume that $BA \subset \lambda AB \neq 0$ where $\lambda \in \mathbb{C}$. Then AB is normal iff $|\lambda| = 1$.*

Proof. First, and since A is closed and B is bounded, AB is automatically closed.

Since A is normal, so is λA . Hence the Fuglede-Putnam (see e.g. [3]) theorem gives

$$BA \subset \lambda AB \implies BA^* \subset \bar{\lambda} A^* B \text{ or } \lambda B^* A \subset AB^*.$$

Using the above "inclusions" we have on the one hand

$$\begin{aligned} (AB)^* AB &\supset B^* A^* AB \\ &= B^* A A^* B \text{ (since } A \text{ is normal)} \\ &\supset \frac{1}{\lambda} B^* A B A^* \\ &\supset \frac{1}{\bar{\lambda} \lambda} B^* B A A^* \\ &= \frac{1}{|\lambda|^2} B^* B A A^*. \end{aligned}$$

Since A and AB are closed, and B is bounded, all of $(AB)^* AB$, $A^* A$ and $B^* B$ are self-adjoint so that "adjointing" the previous inclusion yields

$$(AB)^* AB \subset \frac{1}{|\lambda|^2} A A^* B^* B.$$

As $|\lambda|$ is real, the conditions of Proposition 1.1 are met and we finally obtain

$$(1) \quad (AB)^* AB = \frac{1}{|\lambda|^2} A A^* B^* B.$$

On the other hand, we may write

$$\begin{aligned} AB(AB)^* &\supset ABB^* A^* \\ &= AB^* B A^* \text{ (because } B \text{ is normal)} \\ &\supset \lambda B^* A B A^* \\ &= B^* (\lambda AB) A^* \\ &\supset B^* B A A^*. \end{aligned}$$

As above, we obtain

$$AA^*B^*B \supset AB(AB)^*$$

and by Proposition 1.1 we end up with

$$(2) \quad AB(AB)^* = AA^*B^*B.$$

Accordingly, we clearly see that AB is normal iff $|\lambda| = 1$, completing the proof. \square

Corollary 2.4. *Let A and B be two normal operators where B is bounded. If $BA \subset \lambda AB \neq 0$ where $|\lambda| = 1$, then*

$$\overline{BA} = \lambda AB,$$

where \overline{BA} denotes the closure of the operator BA .

Proof. Since $BA \subset \lambda AB$ and $|\lambda| = 1$, AB is normal by Theorem 2.2.

Since B is bounded and A is densely defined, BA too is densely defined and so it has a unique adjoint. Hence

$$BA \subset \lambda AB \implies \overline{BA}^* A^* = B^* (\lambda A)^* \subset (\lambda AB)^* \subset (BA)^* = A^* B^*$$

or

$$B^* A^* \subset \frac{1}{\overline{\lambda}} A^* B^*.$$

Since B^* is bounded and $|\frac{1}{\overline{\lambda}}| = 1$, Theorem 2.2 applies again and yields the normality of $A^* B^*$. Whence $(\overline{BA})^*$ is normal as $(BA)^* = A^* B^*$ and so $(BA)^{**}$ is normal. Now since λAB is normal, it is closed so that $BA \subset \lambda AB$ tells us that BA is closeable. Therefore, $\overline{BA} = (BA)^{**}$. It follows that \overline{BA} is normal, and that

$$BA \subset \overline{BA} \subset \lambda AB.$$

Finally, as normal operators are maximally normal, we obtain that

$$\overline{BA} = \lambda AB,$$

establishing the result. \square

Proposition 2.2. *Let A and B be two self-adjoint operators where B is bounded. Assume that $BA \subset \lambda AB \neq 0$ where $\lambda \in \mathbb{C}$. Then AB is normal for any λ .*

Proof. Since A and B are self-adjoint, $BA \subset \lambda AB$ implies the following three "inclusions"

$$BA \subset \overline{\lambda} AB, \lambda BA \subset AB \text{ and } \overline{\lambda} BA \subset AB.$$

Proceeding as in the proof of Theorem 2.2, we obtain

$$(AB)^* AB \supset |\lambda|^2 B^2 A^2 \text{ and } AB(AB)^* \supset |\lambda|^2 B^2 A^2.$$

Again as in the proof of Theorem 2.2, AB is normal. \square

Now we apply the foregoing results to give spectral properties of λ -commuting self-adjoint operators (in the unbounded case).

Corollary 2.5. *Let A and B be self-adjoint operators where B is bounded. Assume that $BA \subset \lambda AB \neq 0$ where $\lambda \in \mathbb{C}$. If further $\sigma(B) \cap \sigma(-B) \subseteq \{0\}$, then $\lambda = 1$.*

Remark. The previous result generalizes 2) of Theorem 1.6. Besides the proof of Theorem 1.6 contained a small error which, by the present result, has now been fixed.

Proof. By Proposition 2.2, AB is normal. Thanks to the condition on the spectrum of B and Theorem 1.3 we get that AB is self-adjoint. Hence $(AB)^* = AB$ so that

$$AB = (AB)^* \subset \frac{1}{\lambda}AB.$$

But $D(AB) = D(\alpha AB)$ for any $\alpha \neq 0$. Therefore,

$$AB = \frac{1}{\lambda}AB \text{ or merely } \lambda = 1.$$

□

Corollary 2.6. *Let A and B be self-adjoint operators where B is bounded. Assume that $BA \subset \lambda AB \neq 0$ where $\lambda \in \mathbb{C}$. Then $\lambda = 1$ if B (or $-B$) is positive.*

Corollary 2.7. *Let A and B be self-adjoint operators where B is bounded. Assume that $BA \subset \lambda AB \neq 0$ where $\lambda \in \mathbb{C}$. Then $\lambda \in \{-1, 1\}$.*

Proof. We may write

$$B^2A \subset \lambda BAB \subset \lambda^2 AB^2.$$

Since B is self-adjoint, B^2 is positive so that Corollary 2.6 applies and gives $\lambda^2 = 1$ or $\lambda \in \{-1, 1\}$. □

We finish this paper with the case of two unbounded operators. As should have been expected, this case is quite delicate to handle unless strong assumptions are made. But first, we start by a version of the Fuglede-Putnam theorem.

Theorem 2.3. *Let A , N and M be three unbounded invertible operators on a Hilbert space such that N and M are normal. If $AN = MA$, then*

$$A^*M = NA^* \text{ and } AN^* = M^*A.$$

Proof. We have

$$AN = MA \implies A^{-1}M \subset NA^{-1}.$$

Since A^{-1} is bounded, by Theorem 1.2, we have $A^{-1}M^* \subset N^*A^{-1}$ and hence

$$M^*A \subset AN^*.$$

By taking adjoints (and applying Lemma 1.1) we obtain

$$NA^* \subset A^*M.$$

Now since $A^{-1}M \subset NA^{-1}$, we can get $N^{-1}A^{-1} \subset A^{-1}M^{-1}$. But all operators (in the previous inclusion) are bounded. Therefore, we get

$$N^{-1}A^{-1} = A^{-1}M^{-1}.$$

Applying the Fuglede-Putnam theorem (the all-bounded-operators version) once more yields

$$(N^{-1})^*A^{-1} = A^{-1}(M^{-1})^*.$$

Whence

$$(M^{-1})^*A \subset A(N^{-1})^*, \text{ and thus } AN^* \subset M^*A.$$

By Lemma 1.1 again, $A^*M \subset NA^*$. Thus $AN^* = M^*A$ and $A^*M = NA^*$, establishing the result. □

Corollary 2.8. *If A and B are two unbounded normal and invertible operators such that $AB = \lambda BA$, then*

$$A^*B = \overline{\lambda}BA^* \text{ and } AB^* = \overline{\lambda}B^*A.$$

With Lemma 1.1 and Theorem 2.3 in hand, we may just mimic the proof of Yang-Du (in [19]) to prove the following result:

Theorem 2.4. *Let A, B two unbounded invertible operators such that $AB = \lambda BA \neq 0$, $\lambda \in \mathbb{C}$. Then*

- (1) *If A or B is self-adjoint, then $\lambda \in \mathbb{R}$.*
- (2) *If either A or B is self-adjoint and the other is normal, then $\lambda \in \{-1, 1\}$.*
- (3) *If both A and B are normal, then $|\lambda| = 1$.*

3. A CONJECTURE

In Corollary 2.5 (for example), we said nothing about the spectrum of A . This is in fact due to an (a natural) open question from [9] which the corresponding author of this paper has been working on lately. Let us state it as a conjecture:

Conjecture. *Let A and B be two self-adjoint operators such that only B is bounded and A is positive. Then AB is self-adjoint whenever AB is normal.*

Neither a proof nor a counterexample have been reached yet. However, we can state the following:

- (1) Let $A = -f''$ be defined on $H^2(\mathbb{R})$ (the Sobolev space, which is dense in $L^2(\mathbb{R})$). Then A is an unbounded self-adjoint, and positive operator in $L^2(\mathbb{R})$. Let B be a multiplication operator by an essentially bounded real-valued function φ on \mathbb{R} . Hence B is bounded and self-adjoint on $L^2(\mathbb{R})$.

This "counterexample" does not work for the conditions of the conjecture will force φ to vanish (and so $AB = 0$).

- (2) The conjecture is true with BA in lieu of AB . But, the normality of BA is stronger than that of AB because BA normal will then imply that AB is normal too, and $AB = BA$! Details will appear in another paper.
- (3) The conjecture is true if one assumes further that BA is closed (in such case this will follow from the previous point).
- (4) The conjecture seems to be a hard one. Indeed, the Fuglede (-Putnam) theorem is the tool par excellence when dealing with products involving normal (bounded or unbounded) operators. However, none of the known versions of the Fuglede theorem (such as [6], [9] and [12]) helps us in the proof to get $BA \subset AB$, a sufficient condition to make AB self-adjoint.

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